Abstract

The Loewner property is a quasi-conformal property that can be used to prove Mostow like rigidity of some hyperbolic spaces. Here we present the combinatorial Loewner property (CLP) and boundaries of hyperbolic spaces that satisfy the CLP. This property is a weak version (but conjecturally equivalent) of the Loewner property.

Boundary and rigidity of hyperbolic spaces

Let \( X \) be a hyperbolic space and \( \partial X \) its boundary equipped with a visual metric. Following ideas of Mostow, Gromov, Panz, Bourdon-Pajot we aim to prove the implications below. Let \( f : X \to X \) be a quasi-isometry, then:

\[
\text{if } \partial f \text{ is a quasi-conformal homeo.} \quad \implies \quad \text{Loewner}
\]

\[
\implies \text{(there exists an isometry } F : X \to X \text{ with } \partial F = \partial f.)
\]

Boundaries of hyperbolic spaces that satisfy the Loewner property are rare because they require the knowledge of the conformal dimension.

Combinatorial modulus of curves

Approximation \( G_k \) of \( Z \) a compact metric space. For \( k \in \mathbb{N} \), let \( G_k \) be minimal and homogeneous covering of \( Z \) by quasi-balls of diameter \( 2^{-k} \). \( G_k \)-combinatorial \( p \)-modulus. Let \( \mathcal{F} \neq \emptyset \) be a set of curves of \( Z \). A function \( \rho : G_k \to \mathbb{R}_+ \) is \( \mathcal{F} \)-admissible if:

\[
L_p(\gamma) = \sum_{b \in \mathcal{F}} \rho(b) \geq 1 \text{ for any } \gamma \in \mathcal{F}.
\]

For \( p \geq 1 \) the \( G_k \)-combinatorial \( p \)-modulus of \( \mathcal{F} \) is:

\[
\text{Mod}_{p}(\mathcal{F}, G_k) = \inf \{ \sum_{b \in \mathcal{F}} \rho(b) \},
\]

where the infimum is taken over all the \( \mathcal{F} \)-admissible functions.

Prop.: \( \text{Mod}_{p}(\cdot, G_k) \) is a discrete outer measure on the curves of \( Z \) i.e.

1. if \( \mathcal{F}_1 \subset \mathcal{F}_2 \) then \( \text{Mod}_{p}(\mathcal{F}_1, G_k) \leq \text{Mod}_{p}(\mathcal{F}_2, G_k) \).
2. \( \text{Mod}_{p}(\bigcup_{i=1}^{n} \mathcal{F}_i, G_k) \leq \sum_{i=1}^{n} \text{Mod}_{p}(\mathcal{F}_i, G_k) \).

The CLP

Idea. \( \partial X \) verifies the CLP if the amount of curves joining two continua is controlled by the relative distance between them.

The CLP is easier to prove than the Loewner property: it does not require the knowledge of the conformal dimension and it can be proved geometrically.

Examples

- \( \partial \mathbb{H}^n \) for \( n \geq 3 \) and \( \partial \Delta \) for \( \Delta \) a right-angled Fuchsian building (cf. [2]) satisfy the Loewner property.
- The Sierpiński carpet embedded in \( \mathbb{E}^2 \) satisfies the CLP (cf. [1]). It is conjectured that it is QC to a Loewner spaces.
- Spaces with local cut points do not satisfy the CLP.

A geometric criterion for the CLP

To build a wire netting. \quad \textbf{(S)}: For all \( c > 0 \), there exists \( F \) a finite set of bi-Lipschitz homeo., s.t for any curve \( \gamma \) large enough and any curve \( \gamma' \subset Z \), the set \( \cup_{\gamma \in F} f(\gamma) \) contains a sub-curve \( \gamma' \) with \( d_{\partial F}(\gamma, \gamma') < c \).

According to [1]: \( \textbf{(S)} \implies \text{CLP} \).

A right-angled hyperbolic building

The graph product \( \Gamma \). Let \( D \) be the regular RA dodecahedron in \( \mathbb{H}^3 \) and \( q \in \mathbb{N}, q \geq 3 \). Let \( \{s_1, \ldots, s_{12}\} \) be the faces of \( D \). We define \( \Gamma = \langle s_1, \ldots, s_{12} | s_i^q = 1 \text{ et } [s_i, s_j] = 1 \text{ if } s_i \perp s_j \text{ in } D \rangle \). \( W = \{s_1, \ldots, s_{12} | s_i^q = 1 \text{ et } [s_i, s_j] = 1 \text{ if } s_i \perp s_j \text{ in } D \} \).

The group \( \Gamma \) acts geometrically on \( \Sigma \) a right-angled hyperbolic building of type \( W \) and we identify \( \partial \Sigma \simeq \partial \Gamma \). The Coxeter group \( W \) is the reflection group generated by the faces of \( D \) in \( \mathbb{H}^3 \).

Parabolic Limit Sets

In \( \partial W \) and in \( \partial \Gamma \), parabolic limit sets are obstructions to prove \textbf{(S)}.

In the Coxeter group. If \( M \) is a wall in \( W \) and \( \eta \subset \partial M \), \textbf{(S)} is not verified with \( F \subset W \). However \textbf{(S)} is verified using the symmetries of \( D \).

CLP on \( \partial \Gamma \)

Thanks to \textbf{(S)} in \( \partial W \) and to a control of \( \text{Mod}_{p}(\cdot, G_k) \) in \( \partial \Gamma \) by a combinatorial modulus computed in \( \partial W \), we obtain

Theorem [3]

The boundary \( \partial \Gamma \) equipped with a visual metric satisfies the CLP.

References